

THE GOURSAT PROBLEM FOR THE EINSTEIN-VLASOV SYSTEM: (I) THE INITIAL DATA CONSTRAINTS

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ABSTRACT. We show how to assign, on two intersecting null hypersurfaces, initial data for the Einstein-Vlasov system in harmonic coordinates. As all the components of the metric appear in each component of the stress-energy tensor, the hierarchical method of Rendall can not apply strictly speaking. To overcome this difficulty, an additional assumption have been imposed to the metric on the initial hypersurfaces. Consequently, the distribution function is constrained to satisfy some integral equations on the initial hypersurfaces.

1. INTRODUCTION

This work is devoted to the resolution of the constraints problem associated to the characteristic Einstein-Vlasov (EV) system on two intersecting null hypersurfaces. The interests and physical motivations for studying such problems have been widely mentioned in [2, 4, 5, 6, 10, 12, 13, 14, 17, 18]. It is well known that the EV system is not an evolution system as it stands. In order to obtain a hyperbolic system, one needs to impose some supplementary conditions called gauge conditions which, due to the deep structure of the system, must satisfy the following properties:

(i) whenever these gauge conditions are fulfilled everywhere in the space-time, the EV system reduces to a non-linear hyperbolic system called the evolution system.

(ii) whenever the associated evolution system is satisfied everywhere in the space-time and the gauge conditions are satisfied on the null hypersurfaces that carry the initial data, then these gauge conditions and the complete EV system are satisfied everywhere.

It therefore follows that when the choice of gauge conditions is made, the initial value problem for the EV system is naturally decomposed into two parts called the evolution problem and the constraints problem.

2000 *Mathematics Subject Classification.* Primary 82D05, 83C05; Secondary 34A12, 35L70.

Key words and phrases. Einstein-Vlasov system, Characteristic initial value problem, Null or characteristic hypersurfaces, Harmonic gauge, Gravitational potentials, distribution function.

The resolution of the evolution problem is equivalent to the resolution of the reduced non-linear hyperbolic system obtained from the EV system thanks to the choice of the gauge conditions. Due to the gauge conditions the data for the reduced EV system can not be given freely. It is necessary to construct, from arbitrary choice of some components of the gravitational potentials (called free data) on the initial null hypersurfaces, all the initial data such that the solution of the reduced EV system with those initial data satisfies the gauge conditions on the initial null hypersurfaces. The construction of such data is referred to as the resolution of the constraints problem. Through all the work we will use harmonic gauge for the gravitational field.

We now proceed to survey some relatively recent works known about characteristic initial value problems with initial data prescribed on two intersecting null hypersurfaces often referred to as the Goursat problems. In 1990, A. D. Rendall [17] published a C^∞ existence and uniqueness result for quasilinear hyperbolic systems of second order with C^∞ data prescribed on two intersecting null hypersurfaces. Using the harmonic gauge, the author applied the C^∞ result obtained in [17] to solve the characteristic initial value problem for the Einstein equations in vacuum and with relativistic perfect fluid source. For sake of more physical applications, it is known that, for Partial Differential Equations (PDE), solutions of finite differentiability order are more important than those of infinite differentiability order. In [17] section 7 the author mentioned briefly how results of finite differentiability order can be obtained for data of finite differentiability order although proofs were not given. In 1990, H. Müller zum Hagen [14] used Sobolev type inequalities to derive energy inequalities that enable him solve, in weighted Sobolev space (results of finite differentiability order), the characteristic initial value problem for linear hyperbolic systems of second order. He also predicted an existence and uniqueness result for the quasilinear case. Apart from the fundamental papers [14] and [17], some other works on characteristic initial value problems with initial data prescribed on two intersecting null hypersurfaces can be found in [2, 3, 8, 9, 10, 12, 18]. As pointed out by H. Andreasson [1], A. D. Rendall [17] and M. Fjallborg [11], unlike some known models, the Einstein-Vlasov model has a very nice feature in General Relativity and Kinetic Theory since the stress-energy tensor fulfills, without any supplementary assumption, all the physical necessary energy conditions i.e. the weak energy condition, the dominant energy condition and the strong energy condition as well as the non-negative sum pressures condition. This situation, coupled with the importance of characteristic initial value problems mentioned at the beginning, motivates us to study the constraints problem associated to the characteristic EV system. When attempting to solve the constraints problem for the characteristic EV system by the hierarchical method of Rendall (see [17, 18]), a crucial obstacle occurs due to the complicated form of each component of the stress-energy tensor where all the components of the metric to be constructed appear. The novelty of

our work resides in the fact that we have worked out this difficulty through a supplementary judicious assumption imposed to the gravitational potentials on the initial hypersurfaces. As a consequence of the additional assumption on the gravitational potentials, the distribution function can not be given as free data, it must satisfy some integral equations. Another advantage of this paper is that, unlike the work of Rendall [17, 18], many delicate calculations and expressions are given in details in such a way that we can foresee promising resolution of the global characteristic EV system using for example tools that are similar to those of G. Caciotta and F. Nicolo [3, 4]. To reduce the length of the paper, the evolution problem for the characteristic EV system is out of the scope of the present work and will be solved in a forthcoming paper. The paper is organized as follows. In section 2, we give some preliminaries about the EV system. The complete form as well as the reduced form (in harmonic coordinates) of the EV system are written. A new form of EV system is derived with appropriate unknowns and variables. This new form of EV system is suitable for the resolution of the constraints problem. The concern of section 3 is the resolution of the constraints problem for the characteristic EV system i.e. the construction of the initial data for the reduced EV system such that the harmonic gauge conditions are satisfied on the initial null hypersurfaces. For sake of simplicity and clarity, only the case of C^∞ data will be discussed. Data of finite differentiability order may be constructed in Sobolev type spaces using energy inequalities and other classical tools as described in [9, 10, 14, 17] and references therein. An appendix D is provided at the end of the work and is devoted to the treatment of the constraints integral equations which must be satisfied by the distribution function. It would be of interest to investigate whether the additional assumption on free data as well as the constraints integral equations have a particular physical meaning.

2. THE EINSTEIN-VLASOV (EV) SYSTEM

2.1. The complete form of the EV system. The geometric framework is a four dimensional differentiable manifold \mathcal{M} , endowed with a hyperbolic metric \widehat{g} of signature $-+++$. The manifold $(\mathcal{M}, \widehat{g})$ is called a space-time. \mathcal{M} is assumed to be orientable and of class C^∞ . Throughout the remainder of the work, commas will be used to denote partial derivatives e.g. $\widehat{g}_{ij,k} = \frac{\partial \widehat{g}_{ij}}{\partial y^k}$. Roman indices i, j, \dots run from 1 to 4 while Greek ones α, β, \dots run from 3 to 4. Einstein convention on repeated indices is used i.e. $A_i B^i = \sum_i A_i B^i$. The Einstein-Vlasov system is written as follows (see [1, 7, 11, 19])

$$\begin{aligned} \widehat{S}_{ij} &\equiv \widehat{R}_{ij} - \frac{1}{2} \widehat{R} \widehat{g}_{ij} = \widehat{T}_{ij}, \\ q^i \frac{\partial f}{\partial y^i} - \widehat{\Gamma}_{jk}^i q^j q^k \frac{\partial f}{\partial q^i} &= 0, \end{aligned} \tag{2.1}$$

where \widehat{g}_{ij} are the covariant components of the metric \widehat{g} . They constitute the unknowns for the Einstein equations. \widehat{R}_{ij} are the covariant components of the Ricci tensor and \widehat{R} is the scalar curvature of the metric \widehat{g} . In the local coordinates (y^i) they are given as follows

$$\widehat{R}_{ij} = \widehat{R}_{ikj}^k = \widehat{\Gamma}_{ij,k}^k - \widehat{\Gamma}_{ik,j}^k + \widehat{\Gamma}_{kl}^k \widehat{\Gamma}_{ij}^l - \widehat{\Gamma}_{jl}^k \widehat{\Gamma}_{ik}^l, \quad \widehat{R} = \widehat{g}^{ij} \widehat{R}_{ij}, \quad (2.2)$$

where $\widehat{\Gamma}_{ij}^k$ are the Christoffel symbols of the metric \widehat{g} i.e.,

$$\widehat{\Gamma}_{jk}^i = \frac{1}{2} \widehat{g}^{il} (\widehat{g}_{lk,j} + \widehat{g}_{lj,k} - \widehat{g}_{jk,l}), \quad (2.3)$$

\widehat{g}^{il} are the contravariant components of \widehat{g} i.e.,

$$\widehat{g}^{il} \widehat{g}_{lk} = \delta_k^i = \begin{cases} 1 & \text{for } i = k, \\ 0 & \text{for } i \neq k. \end{cases} \quad (2.4)$$

f is the distribution function (or the particle number density function) which constitutes the unknown for the Vlasov equation. f is a non-negative real valued function defined on $F(\mathcal{M})$, where

$$F(\mathcal{M}) = \bigcup_{y \in \mathcal{M}} \{q = (q^i) \in T_y : \widehat{g}_{ij}(y) q^i q^j = -m^2, \ 0 < q^1\}, \quad (2.5)$$

with $T_y \equiv T_y \mathcal{M}$. The Vlasov equation symbolizes the conservation of the number of particles along the trajectories across the hypersurfaces of $F(\mathcal{M})$ in the case where there is no collision between particles. \widehat{T}_{ij} are the covariant components of the stress-energy (or energy-momentum) tensor which is the source of the gravitational field created by the particles. In contravariant components the stress-energy is defined by the following relation (see [7])

$$\widehat{T}^{ij}(y) = - \int_{F_y} f(y, q) q^i q^j \frac{|\widehat{g}|^{\frac{1}{2}}}{q_1} d^3 q, \quad (2.6)$$

where $F_y = \{q = (q^i) \in T_y : \widehat{g}_{ij}(y) q^i q^j = -m^2, \ 0 < q^1\}$, $d^3 q = dq^2 \wedge dq^3 \wedge dq^4$, $|\widehat{g}|$ is the modulus of the determinant of (\widehat{g}_{ij}) .

2.2. The reduced EV system. The Einstein equations as they stand are not hyperbolic but in harmonic coordinates they read (see [7])

$$\widehat{R}_{ij}^h = \widehat{T}_{ij}, \quad (2.7)$$

where

$$\widehat{R}_{ij}^h \equiv \widehat{R}_{ij} - \frac{1}{2} \left(\widehat{g}_{ik} \widehat{\Gamma}_{,j}^k + \widehat{g}_{jk} \widehat{\Gamma}_{,i}^k \right) = -\frac{1}{2} \widehat{g}^{km} \widehat{g}_{ij,mk} + Q_{ij}. \quad (2.8)$$

Here Q_{ij} is a rational function depending on the metric components and their first order derivatives (see [10, 20]),

$$\widehat{\Gamma}^k = \widehat{g}^{ij} \widehat{\Gamma}_{ij}^k. \quad (2.9)$$

So the reduced EV system reads

$$\begin{aligned} -\frac{1}{2}\widehat{g}^{km}\widehat{g}_{ij,mk} + Q_{ij} &= \widehat{T}_{ij}, \\ q^i \frac{\partial f}{\partial y^i} + Q^i \frac{\partial f}{\partial q^i} &= 0, \end{aligned} \quad (2.10)$$

where

$$Q^i = -\widehat{\Gamma}_{jk}^i q^j q^k. \quad (2.11)$$

2.3. Appropriate unknowns and variables. As a relativistic speed is bounded, we think that it is convenient to choose on the mass shell, coordinates with bounded domain (see also [7]). Let $y \in \mathbb{U}$, \mathbb{U} is the domain of a local chart in \mathcal{M} . Set $w^A = \frac{q^A}{q^1}$, $A = 2, 3, 4$ and denote by M_y the image in \mathbb{R}^3 of F_y by the mapping $(q^i) \mapsto (w^A)$. Assume the following hyperbolicity conditions on (\widehat{g}_{ij}) .

Assumption \widehat{h} : The metric (\widehat{g}_{ij}) is uniformly hyperbolic and the hypersurfaces $y^1 = Cte$ are uniformly spatial i.e.

$$\begin{aligned} \exists a, b \in (0, \infty) : a^2 |\xi|^2 &\leq \widehat{g}_{AB} \xi^A \xi^B \leq b^2 |\xi|^2 \text{ where } |\xi|^2 = \sum_{A=2}^4 (\xi^A)^2, \\ -\widehat{g}_{11} &\geq a^2 \text{ and } -\widehat{g}^{11} \geq a^2. \end{aligned} \quad (2.12)$$

Proposition 1. (i) Under assumption (2.12), the Vlasov equation reads

$$q^i \frac{\partial f}{\partial y^i} + Q^A \frac{\partial f}{\partial q^A} = 0. \quad (2.13)$$

(ii) Under assumption (2.12), M_y is a bounded domain in \mathbb{R}^3 such that $M_y \subset M$, where M is a fixed compact domain in \mathbb{R}^3 . The stress-energy tensor is given as follows

$$\widehat{T}^{ij}(y) = \frac{1}{m^2} \int_{M_y} f(y, w) q^i q^j (q^1)^4 |\widehat{g}|^{\frac{1}{2}} d^3 w, \quad (2.14)$$

where $d^3 w = dw^2 \wedge dw^3 \wedge dw^4$, $f(y, w)$ is the expression of $f(y, q)$ in the local coordinates (y, w) .

Proof. See [7]. □

Remark 1. In the expression (2.14) of the stress-energy tensor, we would like to write $f(y, w) q^i q^j (q^1)^4$ as $\varphi(y, w) w^{ij}$, where w^{ij} does not depend on (\widehat{g}_{ij}) . To do so, we proceed to the following change of the unknown distribution function by setting $f(y, w) = \varphi(y, w) (q^1)^{-6}$. So we must have $w^{ij} = \frac{q^i q^j}{(q^1)^2}$.

Proposition 2. Under the change $f(y, w) = \varphi(y, w) (q^1)^{-6}$, the stress-energy tensor is given as follows

$$\widehat{T}^{ij}(y) = \frac{1}{m^2} \int_{M_y} \varphi(y, w) w^{ij} |\widehat{g}|^{\frac{1}{2}} d^3 w, \quad (2.15)$$

where

$$w^{ij} = \frac{q^i q^j}{(q^1)^2}.$$

The Vlasov equation becomes

$$q^i \frac{\partial \varphi}{\partial y^i} + \frac{1}{q^1} (Q^A - w^A Q^1) \frac{\partial \varphi}{\partial w^A} - \frac{6}{q^1} Q^1 \varphi = 0. \quad (2.16)$$

Proof. See [7]. \square

Remark 2. The expression (2.15) of the stress-energy tensor is not appropriate since the domain M_y depends on y and makes it difficult to differentiate \widehat{T}^{ij} even in the distributional sense. It appears therefore judicious to transform this domain in order to make it independent of y .

Assume the following decomposition of \widehat{g} .

Assumption \widehat{h}' : The spatial part of (\widehat{g}_{AB}) is decomposed as follows

$$\widehat{g}_{AB} Y^A Y^B = \sum_{B=2}^4 (\lambda_A^B Y^A)^2, \quad (2.17)$$

where λ_A^B are functions that depends smoothly (C^∞ for instance) on the components \widehat{g}_{AB} of the metric. Set

$$v^C = (-\widehat{g}^{11})^{\frac{1}{2}} \lambda_A^C [w^A + \widetilde{g}^{1A}], \quad (2.18)$$

with

$$\widetilde{g}^{1A} = -\frac{1}{\widehat{g}^{11}} \widehat{g}^{1A}. \quad (2.19)$$

(2.18) is equivalent to the following relation

$$w^A = (\lambda_B^A)^{-1} v^B (-\widehat{g}^{11})^{-\frac{1}{2}} - \widetilde{g}^{1A}, \quad (2.20)$$

where $(\lambda_B^A)^{-1}$ are the components of the inverse of the matrix (λ_J^I) .

Proposition 3. The image of M_y by the mapping $(w^A) \mapsto (v^A)$ is the unit open ball B in \mathbb{R}^3 . In the parameters (v^A) , the energy-momentum tensor reads

$$\widehat{T}^{ij}(y) = \frac{1}{m^2} \int_B \varphi(y, v) v^{ij} |\widehat{g}|^{\frac{1}{2}} (-\widehat{g}^{11})^{-\frac{3}{2}} |\widetilde{g}|^{-\frac{1}{2}} d^3 v, \quad (2.21)$$

where $d^3 v = dv^2 \wedge dv^3 \wedge dv^4$, $v^{ij} = \frac{q^i q^j}{(q^1)^2}$, $|\widetilde{g}|$ is the modulus of the determinant of (\widehat{g}_{AB}) , $\varphi(y, v)$ is the expression of $\varphi(y, w)$ in the local coordinates (y, v) .

The Vlasov equation reads as follows

$$\begin{aligned} \frac{\partial \varphi}{\partial y^1} + \left[(\lambda_B^A)^{-1} v^B (-\widehat{g}^{11})^{-\frac{1}{2}} - \widehat{g}^{1A} \right] \frac{\partial \varphi}{\partial y^A} \\ + (q^1)^{-2} (-\widehat{g}^{11})^{\frac{1}{2}} \lambda_B^A (Q^B - w^B Q^1) \frac{\partial \varphi}{\partial v^A} - 6 (q^1)^{-2} Q^1 \varphi = 0. \end{aligned} \quad (2.22)$$

Proof. See [7]. \square

Remark 3. *Y. Choquet-Bruhat [7] used assumption (2.17) and a variant of assumption (2.12) to treat the ordinary Cauchy problem for the EV system. But in the characteristic case, those assumptions are not appropriate and they need to be recast. We proceed to the desired adaptation through a judicious change of local events variables (y^i) .*

Proposition 4. *Let (y^i) be a local coordinates system on \mathcal{M} in which the components (\widehat{g}_{ij}) of the metric satisfy assumption (2.12). Set*

$$\begin{aligned} x^1 &= \frac{1}{2} (y^1 + y^2), \quad x^2 = \frac{1}{2} (y^1 - y^2), \\ x^\alpha &= y^\alpha, \quad \alpha = 3, 4. \end{aligned} \quad (2.23)$$

In the coordinates system (x, p) , the EV system reads

$$\begin{aligned} S_{ij} &\equiv R_{ij} - \frac{1}{2} R g_{ij} = T_{ij}, \\ p^i \frac{\partial f}{\partial x^i} + P^i \frac{\partial f}{\partial p^i} &= 0, \end{aligned} \quad (2.24)$$

where

$$\begin{aligned} g_{ij}(x) &= \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j} \widehat{g}_{kl}(y), \quad R_{ij}(x) = \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j} \widehat{R}_{kl}(y), \\ R(x) &= g^{ij}(x) R_{ij}(x), \quad T_{ij}(x) = \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j} \widehat{T}_{kl}(y), \\ p^i &= \frac{\partial x^i}{\partial y^k} q^k, \quad P^i = \frac{\partial x^i}{\partial y^k} Q^k = -\Gamma_{jk}^i p^j p^k, \\ \Gamma_{jk}^i &= \frac{1}{2} g^{ic} (g_{ck,j} + g_{cj,k} - g_{jk,c}). \end{aligned} \quad (2.25)$$

Proof. A direct calculation leads to the desired equations. \square

Remark 4. *The change of local coordinates (2.23) preserves the harmonicity. In other words, we have the following equivalence*

$$\left(\forall i = 1, 2, 3, 4, \widehat{\Gamma}^i \equiv \widehat{g}^{kl} \widehat{\Gamma}_{kl}^i = 0 \right) \Leftrightarrow \left(\forall i = 1, 2, 3, 4, \Gamma^i \equiv g^{kl} \Gamma_{kl}^i = 0 \right). \quad (2.26)$$

In fact, the following relations hold

$$\begin{aligned} \widehat{\Gamma}^1 &= \Gamma^1 + \Gamma^2, \quad \widehat{\Gamma}^2 = \Gamma^1 - \Gamma^2, \\ \widehat{\Gamma}^\alpha &= \Gamma^\alpha, \quad \alpha = 3, 4. \end{aligned} \quad (2.27)$$

Proposition 5. *The stress-energy tensor (2.21) is given in the local coordinates (x^i) and the local parameters (v^A) as follows*

$$T^{ij}(x) = \frac{1}{2m^2} \int_B \varphi(x, v) v^{ij} |g|^{\frac{1}{2}} (-\widehat{g}^{11})^{-\frac{3}{2}} |\widetilde{g}|^{-\frac{1}{2}} d^3 v, \quad (2.28)$$

where $v^{ij} = \frac{p^i p^j}{(q^1)^2}$, $p^i = \frac{\partial x^i}{\partial y^k} q^k$, $|g|$ is the modulus of the determinant of (g_{ij}) , $|\tilde{g}|$ is the modulus of the determinant of (\hat{g}_{AB}) , $\varphi(x, v)$ is the expression of $\varphi(y, v)$ in the local coordinates (x, v) . The Vlasov equation (2.22) becomes

$$H^i \frac{\partial \varphi}{\partial x^i} + L^C \frac{\partial \varphi}{\partial v^C} + F\varphi = 0, \quad (2.29)$$

with

$$\begin{aligned} w^A &= (\lambda_B^A)^{-1} v^B (-\hat{g}^{11})^{-\frac{1}{2}} - \tilde{g}^{1A}, \quad \hat{g}^{11}(y) = g^{11}(x) + 2g^{12}(x) + g^{22}(x), \\ H^1(x, v) &= \frac{1}{2}(1 + w^2), \quad H^2(x, v) = \frac{1}{2}(1 - w^2), \quad H^\alpha(x, v) = w^\alpha, \quad \alpha = 3, 4, \\ L^C(x, v) &= \frac{1}{4}(-\hat{g}^{11})^{\frac{1}{2}} \lambda_2^C l^2(x, v) - \frac{1}{4}(-\hat{g}^{11})^{\frac{1}{2}} \lambda_\alpha^C l^\alpha(x, v), \quad F(x, v) = \frac{3}{2}w^2 \tilde{F}. \end{aligned} \quad (2.30)$$

Here

$$\begin{aligned} \tilde{F} &= (\Gamma_{11}^2 + \Gamma_{11}^1)(1 + w^2)^2 + (\Gamma_{22}^2 + \Gamma_{22}^1)(1 - w^2)^2 + 2(\Gamma_{12}^2 + \Gamma_{12}^1)(1 - (w^2)^2) \\ &\quad + 4(\Gamma_{1\lambda}^2 + \Gamma_{1\lambda}^1)w^\lambda(1 + w^2) + 4(\Gamma_{2\lambda}^2 + \Gamma_{2\lambda}^1)w^\lambda(1 - w^2) + 4(\Gamma_{\lambda\mu}^2 + \Gamma_{\lambda\mu}^1)w^\lambda w^\mu, \end{aligned}$$

and

$$l^2(x, v) = l_1^2 + w^2 l_2^2, \quad l^\alpha(x, v) = l_1^\alpha - w^\alpha l_2^2, \quad \alpha = 3, 4,$$

with

$$\begin{aligned} l_1^2 &= (\Gamma_{11}^2 - \Gamma_{11}^1)(1 + w^2)^2 + (\Gamma_{22}^2 - \Gamma_{22}^1)(1 - w^2)^2 \\ &\quad + 2(\Gamma_{12}^2 - \Gamma_{12}^1)(1 - (w^2)^2) + 4(\Gamma_{1\lambda}^2 - \Gamma_{1\lambda}^1)w^\lambda(1 + w^2) \\ &\quad + 4(\Gamma_{2\lambda}^2 - \Gamma_{2\lambda}^1)w^\lambda(1 - w^2) + 4(\Gamma_{\lambda\mu}^2 - \Gamma_{\lambda\mu}^1)w^\lambda w^\mu, \\ l_2^2 &= (\Gamma_{11}^2 + \Gamma_{11}^1)(1 + w^2)^2 + (\Gamma_{22}^2 + \Gamma_{22}^1)(1 - w^2)^2 \\ &\quad + 2(\Gamma_{12}^2 + \Gamma_{12}^1)(1 - (w^2)^2) + 4(\Gamma_{1\lambda}^2 + \Gamma_{1\lambda}^1)w^\lambda(1 + w^2) \\ &\quad + 4(\Gamma_{2\lambda}^2 + \Gamma_{2\lambda}^1)w^\lambda(1 - w^2) + 4(\Gamma_{\lambda\mu}^2 + \Gamma_{\lambda\mu}^1)w^\lambda w^\mu, \\ l_1^\alpha &= \Gamma_{11}^\alpha(1 + w^2)^2 + \Gamma_{22}^\alpha(1 - w^2)^2 \\ &\quad + 2\Gamma_{12}^\alpha(1 - (w^2)^2) + 4\Gamma_{1\lambda}^\alpha w^\lambda(1 + w^2) \\ &\quad + 4\Gamma_{2\lambda}^\alpha w^\lambda(1 - w^2) + 4\Gamma_{\lambda\mu}^\alpha w^\lambda w^\mu. \end{aligned}$$

Proof. It is straightforward though lengthy. \square

Remark 5. The EV system (2.24) in the local coordinates (x, v) reads as follows

$$R_{ij} - \frac{1}{2}Rg_{ij} = T_{ij}, \quad H^i \frac{\partial \varphi}{\partial x^i} + L^C \frac{\partial \varphi}{\partial v^C} + F\varphi = 0. \quad (2.31)$$

The reduced EV system (2.10) in the local coordinates (x, v) reads as follows

$$\tilde{R}_{ij} = T_{ij}, \quad H^i \frac{\partial \varphi}{\partial x^i} + L^C \frac{\partial \varphi}{\partial v^C} + F\varphi = 0, \quad (2.32)$$

where

$$\tilde{R}_{ij} \equiv R_{ij} - \frac{1}{2} (g_{ki} \Gamma_{,j}^k + g_{kj} \Gamma_{,i}^k) = -\frac{1}{2} g^{km} g_{ij, mk} + Q_{ij}. \quad (2.33)$$

3. THE CONSTRAINTS PROBLEM FOR THE EV SYSTEM

The task here is the construction of initial data for the reduced EV such that the constraints $\Gamma^k = 0$ are satisfied on $G^1 \cup G^2$, where G^1 and G^2 are the hypersurfaces in \mathbb{R}^4 defined by $x^1 = 0$ and $x^2 = 0$ respectively. We will need $G_T^\omega = \{x \in G^\omega : 0 \leq x^1 + x^2 \leq T\}$, $T > 0$, $\omega = 1, 2$. Here the problem is much more difficult than in [10, 17]. This difficulty has something to do with the appearance of all the components of the metric in any component of the stress-energy tensor. To overcome this toughness, we add a supplementary assumption on the metric along the initial hypersurfaces. All the same, we try to mimic, as far as possible, the hierarchical method of Rendall [17, 18]. We will see that the supplementary assumption has as consequence to force the distribution function to satisfy specific integral equations on the initial hypersurfaces. The resolution of the integral equations derived can be achieved under suitable conditions (see appendix D). Nevertheless it is still to be investigated whether our additional assumption has a particular physical meaning. The construction will be made in a standard harmonic coordinates system. The existence of such standard harmonic coordinates system has been established by A. D. Rendall [17].

Let us now adapt the method of Rendall to construct C^∞ initial data for the EV system. The assumptions under which the work is achieved are such that only the data $g_{\alpha\beta}$ and g_{12} have to be constructed on $G^1 \cup G^2$, the relations $\Gamma^k = 0$ have to be arranged on $G^1 \cup G^2$, the relations $g_{22,1} = 2g_{12,2}$ and $g_{11,2} = 2g_{12,1}$ have to be established on G^1 and G^2 respectively. The construction of the data is done fully on G^1 and it will be clear that data on G^2 are constructed in quite a similar way. The first level of the hierarchy is now described.

3.1. Construction of $g_{\alpha\beta}$ and g_{12} on G_T^1 , relations $\Gamma^1 = 0$ and $g_{22,1} = 2g_{12,2}$. Let $T \in (0, \infty)$, $(h_{\alpha\beta}) = \begin{bmatrix} h_{33} & h_{34} \\ h_{34} & h_{44} \end{bmatrix}$ a matrix with determinant 1 at each point. Set $g_{\alpha\beta} = \Omega h_{\alpha\beta}$, where $\Omega > 0$ is an unknown function called the conformity factor. Assume as in [10, 17] that

$$g_{22} = g_{23} = g_{24} = 0 \text{ on } G_T^1. \quad (3.1)$$

The additional assumption on free data is the following

$$g_{11} = g_{13} = g_{14} = 0 \text{ on } G_T^1. \quad (3.2)$$

On G_T^1 it holds

$$\begin{aligned} g_{12} g^{12} &= 1, & g^{11} &= g^{1\alpha} = 0, \\ g^{22} &= g^{2\alpha} = 0, & g_{\lambda\beta} g^{\alpha\beta} &= \delta_\lambda^\alpha. \end{aligned} \quad (3.3)$$

3.1.1. Expression of R_{22} and T_{22} .

Proposition 6. *On G_T^1 , it holds that*

$$\begin{aligned} R_{22} &= \frac{1}{4}g^{12}g^{\alpha\beta}g_{\alpha\beta,2}(2g_{12,2} - g_{22,1}) + \frac{1}{4}g_{,2}^{\beta\lambda}g_{\lambda\beta,2} - \frac{1}{2}(g^{\alpha\beta}g_{\alpha\beta,2})_{,2}, \\ T_{22} &= (g_{12})^4 K_{22}, \end{aligned} \quad (3.4)$$

where

$$K_{22}(x) = \frac{1}{16m^2} \int_B \varphi(x, v) (1 + v^2)^2 d^3v. \quad (3.5)$$

Proof. See appendix A. \square

Assume $g_{2i} = 0$ for $i \neq 1$ on G_T^1 , $g_{22,1} = 2g_{12,2}$ on G_T^1 . Then $\Gamma^1 = 0$ is equivalent to (see [10, 17])

$$g_{12,2} = \frac{1}{2}g_{12} \frac{\Omega_{,2}}{\Omega}. \quad (3.6)$$

The equation

$$\frac{1}{4}g_{,2}^{\alpha\beta}g_{\alpha\beta,2} - \frac{1}{2}(g^{\alpha\beta}g_{\alpha\beta,2})_{,2} = T_{22}, \quad (3.7)$$

provides the following non linear second order ODE with the conformity factor Ω as unknown

$$-\left(\frac{\Omega_{,2}}{\Omega}\right)^2 + \frac{1}{2}h_{\alpha\beta,2}h_{,2}^{\alpha\beta} - 2\left(\frac{\Omega_{,2}}{\Omega}\right)_{,2} = 2K_{22}(g_{12})^4. \quad (3.8)$$

If we set $\Omega = e^V$, then the following system of ODE is derived from (3.6) and (3.8) in order to determine the conformity factor together with g_{12}

$$\begin{aligned} 2V_{,22} &= -(V_{,2})^2 - 2K_{22}(g_{12})^4 + \frac{1}{2}h_{\alpha\beta,2}h_{,2}^{\alpha\beta}, \\ g_{12,2} &= \frac{1}{2}g_{12}V_{,2}. \end{aligned} \quad (3.9)$$

Let $T \in (0, \infty)$. Assume $h_{\alpha\beta}, K_{22} \in C^\infty(G_T^1)$. Take $V_0, V_1, W_0 \in C^\infty(\Gamma)$, where $\Gamma \equiv G_T^1 \cap G_T^2$. Then there exists $T_1 \in]0, T]$ such that (3.9) has a unique solution $(V, g_{12}) \in C^\infty(G_{T_1}^1) \times C^\infty(G_{T_1}^1)$ satisfying $V = V_0, V_{,2} = V_1, g_{12} = W_0$ on Γ . This follows from known local existence and uniqueness results concerning non-linear ODE with C^∞ data in Banach spaces.

3.1.2. *The condition $g_{22,1} - 2g_{12,2} = 0$ on $G_{T_1}^1$.* On $G_{T_1}^1$, the reduced equation $\tilde{R}_{22} = T_{22}$ is equivalent to the following homogenous ODE with unknown $g_{22,1} - 2g_{12,2}$ (see [10, 17])

$$(g^{12})^2 g_{12,2} (g_{22,1} - 2g_{12,2}) - g^{12} (g_{22,1} - 2g_{12,2})_{,2} = 0. \quad (3.10)$$

Assume $g_{22,1} - 2g_{12,2} = 0$ on Γ . Then $g_{22,1} - 2g_{12,2} = 0$ on $G_{T_1}^1$ and so $\Gamma^1 = 0$ on $G_{T_1}^1$.

3.2. Relations $\Gamma^\alpha = 0$ on $G_{T_1}^1$. We seek for a combination between $R_{2\alpha}$ and Γ^α that will provide an homogenous ODE on G^1 with unknown Γ^α .

Proposition 7. *On G_T^1 , it holds that*

$$R_{2\alpha} + \frac{1}{2}g_{\alpha\beta}\Gamma_{,2}^\beta + \left(g^{12}g_{12,2}g_{\alpha\beta} + \frac{1}{2}g_{\alpha\beta,2}\right)\Gamma^\beta = \psi_\alpha, \quad (3.11)$$

$$T_{2\alpha} = -\frac{(-g_{12})^{\frac{7}{2}}h_{\alpha 3}\Omega^{\frac{1}{2}}(h_{33})^{-\frac{1}{2}}}{8\sqrt{2}m^2}K_3 - \frac{(-g_{12})^{\frac{7}{2}}\left(h_{\alpha 4}\Omega^{\frac{1}{2}}(h_{33})^{\frac{1}{2}} - \Omega h_{\alpha 3}h_{34}\right)}{8\sqrt{2}m^2}K_4, \quad (3.12)$$

where

$$\begin{aligned} \psi_\alpha = & \frac{1}{2}(g^{12})g_{12,2}\left[-2g^{12}g_{12,\alpha} + g^{\mu\theta}(2g_{\alpha\mu,\theta} - g_{\mu\theta,\alpha})\right] \\ & + \frac{1}{4}g_{\alpha\beta,2}\left[-2g^{\beta\lambda}g^{12}g_{12,\lambda} + g^{\beta\lambda}g^{\mu\theta}(2g_{\lambda\mu,\theta} - g_{\mu\theta,\lambda})\right] \\ & + \frac{1}{2}(g^{\lambda\beta}g_{\alpha\beta,2})_{,\lambda} - 3(g^{12}g_{12,2})_{,\alpha} - \frac{3}{2}(g^{12})^2g_{12,2}g_{12,\alpha} \\ & + \frac{1}{2}g^{12}(g_{22,1\alpha} + g_{12,2\alpha}) \\ & + \frac{1}{2}g_{,2}^{\beta\lambda}(g_{\lambda\beta,\alpha} + g_{\lambda\alpha,\beta}) + (g^{12})^2g_{12,2}g_{12,\alpha} \\ & + \frac{1}{2}g_{\alpha\beta}\left[-2g^{\beta\lambda}g^{12}g_{12,\lambda} + g^{\beta\lambda}g^{\mu\theta}(2g_{\lambda\mu,\theta} - g_{\mu\theta,\lambda})\right]_{,2}, \\ K_3 = & \int_B \varphi(x, v)(1 + v^2)v^3d^3v, \quad K_4 = \int_B \varphi(x, v)(1 + v^2)v^4d^3v. \end{aligned} \quad (3.13)$$

Proof. See appendix B. \square

The relations $\Gamma^3 = \Gamma^4 = 0$ on G^1 is to be arranged under a suitable choice of the distribution function on $\widehat{G^1} = G^1 \times B$. It is at this level that the Rendall method need to be modified. Assume that the distribution function φ is such that

$$T_{2\alpha} = \psi_\alpha \text{ on } G_{T_1}^1. \quad (3.14)$$

Then the reduced system $\widetilde{R}_{2\alpha} = T_{2\alpha}$ is equivalent to the following homogenous system of ODE on $G_{T_1}^1$ with unknown (Γ^3, Γ^4)

$$\begin{aligned} g_{3\beta}\Gamma_{,2}^\beta + \left(g^{12}g_{12,2}g_{3\beta} + \frac{1}{2}g_{3\beta,2}\right)\Gamma^\beta &= 0, \\ g_{4\beta}\Gamma_{,2}^\beta + \left(g^{12}g_{12,2}g_{4\beta} + \frac{1}{2}g_{4\beta,2}\right)\Gamma^\beta &= 0. \end{aligned} \quad (3.15)$$

Assumption (3.14) is an integral system on $G_{T_1}^1$ in the sense that it is written explicitly as follows

$$A_\alpha U + B_\alpha V = \psi_\alpha, \quad (3.16)$$

where

$$\begin{aligned} A_\alpha = & -\frac{(-g_{12})^{\frac{7}{2}}h_{\alpha 3}\Omega^{\frac{1}{2}}(h_{33})^{-\frac{1}{2}}}{8\sqrt{2}m^2}, \quad B_\alpha = -\frac{(-g_{12})^{\frac{7}{2}}\Omega^{\frac{1}{2}}\left(h_{\alpha 4}(h_{33})^{\frac{1}{2}} - \Omega^{\frac{1}{2}}h_{\alpha 3}h_{34}\right)}{8\sqrt{2}m^2} \\ U = & \int_B \varphi(x, v)(1 + v^2)v^3d^3v, \quad V = \int_B \varphi(x, v)(1 + v^2)v^4d^3v. \end{aligned} \quad (3.17)$$

The determinant of the system (3.16) is equal to $\frac{(-g_{12})^7\Omega}{128m^4}$ on $G_{T_1}^1$. So the solutions are given by the relations below

$$\begin{aligned} U &= \frac{128m^4}{(g_{12})^7\Omega} (\psi_4 B_3 - \psi_3 B_4), \\ V &= \frac{128m^4}{(g_{12})^7\Omega} (\psi_3 A_4 - \psi_4 A_3). \end{aligned} \quad (3.18)$$

In brief φ has to be chosen in such a way that the following integral system (IS) holds for every $x \in G_{T_1}^1$

$$\begin{aligned} \int_B \varphi(x, v) (1 + v^2) v^3 d^3 v &= \frac{128m^4}{(g_{12})^7\Omega} (\psi_4 B_3 - \psi_3 B_4), \\ \int_B \varphi(x, v) (1 + v^2) v^4 d^3 v &= \frac{128m^4}{(g_{12})^7\Omega} (\psi_3 A_4 - \psi_4 A_3). \end{aligned} \quad (\text{IS})$$

Assume $\Gamma^\beta = 0$ on Γ . Then, in view of (3.15), $\Gamma^\beta = 0$ on $G_{T_1}^1$.

3.3. Relation $\Gamma^2 = 0$ on $G_{T_1}^1$. We seek for a combination of $g^{\alpha\beta} R_{\alpha\beta}$, Γ^2 and $\Gamma_{,2}^2$ that will provide an homogenous ODE on $G_{T_1}^1$ with unknown Γ^2 .

3.3.1. Combination of $g^{\alpha\beta} R_{\alpha\beta}$, Γ^2 and $\Gamma_{,2}^2$.

Proposition 8. *On $G_{T_1}^1$ the following combination holds*

$$g^{\alpha\beta} R_{\alpha\beta} - 2\Gamma_{,2}^2 - 2g^{12} g_{12,2} \Gamma^2 = \frac{1}{4} g^{\alpha\beta} (N_{\alpha\beta} + M_{\alpha\beta}), \quad (3.21)$$

where

$$\begin{aligned} N_{\alpha\beta} &= -g^{12} g^{\lambda\mu} g_{2\lambda,1} (g_{\beta\mu,\alpha} + g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu}) \\ &\quad + g^{12} (g_{2\beta,1\alpha} + g_{2\alpha,1\beta}) - [2g^{12} g_{12,\alpha} + g^{\lambda\mu} (g_{\mu\lambda,\alpha} + g_{\mu\alpha,\lambda} - g_{\alpha\lambda,\mu})]_{,\beta}, \end{aligned} \quad (3.22)$$

$$\begin{aligned} M_{\alpha\beta} &= [2g^{12} g_{12,\lambda} + g^{\mu\theta} (g_{\mu\theta,\lambda} + g_{\theta\lambda,\mu} - g_{\mu\lambda,\theta})] [g^{\mu\lambda} (g_{\mu\beta,\alpha} + g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu})] \\ &\quad - (g^{12})^2 (g_{12,\beta} + g_{2\beta,1}) (g_{12,\alpha} + g_{2\alpha,1}) \\ &\quad - [g^{12} (g_{12,\beta} - g_{2\beta,1})] [g^{12} (g_{12,\alpha} - g_{2\alpha,1})] \\ &\quad - [g^{\theta\mu} (g_{\theta\lambda,\beta} + g_{\theta\beta,\lambda} - g_{\lambda\beta,\theta})] [g^{\delta\lambda} (g_{\delta\mu,\alpha} + g_{\delta\alpha,\mu} - g_{\alpha\mu,\delta})]. \end{aligned} \quad (3.23)$$

Proof. It follows from relations (7.58 – 61) of [10] by using (3.2) and (3.3). \square

3.3.2. Expression of $g^{\alpha\beta} T_{\alpha\beta}$.

Proposition 9. *On $G_{T_1}^1$ it holds that*

$$\begin{aligned} g^{\alpha\beta} T_{\alpha\beta} &= \frac{(-g_{12})^3}{8m^2} \int_B \varphi(x, v) (v^3)^2 d^3 v \\ &\quad + \frac{(-g_{12})^3 h_{34}}{4m^2} \left(1 - (\Omega h_{33})^{\frac{1}{2}}\right) \int_B \varphi(x, v) v^3 v^4 d^3 v \\ &\quad + \frac{(-g_{12})^3}{8m^2} \left[(h_{34})^2 (\Omega h_{33})^{\frac{1}{2}} \left((\Omega h_{33})^{\frac{1}{2}} - 2 \right) + h_{44} h_{33} \right] \int_B \varphi(x, v) (v^4)^2 d^3 v. \end{aligned} \quad (3.24)$$

Proof. See appendix C. \square

In addition to assumption (3.14), assume

$$g^{\alpha\beta} T_{\alpha\beta} = \frac{1}{4} g^{\alpha\beta} (N_{\alpha\beta} + M_{\alpha\beta}) \text{ on } G_{T_1}^1. \quad (3.25)$$

Then, in view of (3.21), the reduced system $\tilde{R}_{\alpha\beta} = T_{\alpha\beta}$ provides the following homogenous ODE on $G_{T_1}^1$ with unknown Γ^2

$$-2\Gamma_{,2}^2 - 2g^{12}g_{12,2}\Gamma^2 = 0. \quad (3.26)$$

The assumption (3.25) is a supplementary integral equation which is written explicitly as follows, $\forall x \in G_{T_1}^1$

$$\begin{aligned} & \frac{(-g_{12})^3}{8m^2} \int_B \varphi(x, v) (v^3)^2 d^3v + \frac{(-g_{12})^3 h_{34}}{4m^2} \left(1 - (\Omega h_{33})^{\frac{1}{2}}\right) \int_B \varphi(x, v) v^3 v^4 d^3v \\ & + \frac{(-g_{12})^3}{8m^2} \left[(h_{34})^2 (\Omega h_{33})^{\frac{1}{2}} \left((\Omega h_{33})^{\frac{1}{2}} - 2 \right) + h_{44} h_{33} \right] \int_B \varphi(x, v) (v^4)^2 d^3v \\ & = \frac{1}{4} g^{\alpha\beta} (N_{\alpha\beta} + M_{\alpha\beta}). \end{aligned} \quad (\text{IE})$$

In view of (3.26), assuming $\Gamma^2 = 0$ on Γ gives $\Gamma^2 = 0$ on $G_{T_1}^1$.

What we have just proved for the resolution of the constraints problem associated to the EV system can be summed up in the following main theorem.

Theorem 1. *Under the suitable integral assumptions (IS) and (IE) on the distribution function, there exists initial data for the reduced EV system such that the constraints $\Gamma^k = 0$ are satisfied on $G^1 \cup G^2$ for the corresponding solution of the evolution problem associated to the EV system.*

Remark 6. *It would be interesting to see whether the constraints integral equations can be avoided. One way of doing this is to work in temporal gauge and null moving frame (see [15]). But in the harmonic gauge case the issue may not be evident. Nevertheless we think that one could use an orthonormal frame in order to avoid that the metric appears (in an involved way) in the energy-momentum tensor (see [16]).*

APPENDIX A: PROOF OF PROPOSITION 6

The first equality of (3.4) is provided in relation (7.42) of [10]. We handle the second one by using the expression (2.28) of the energy-momentum tensor given in Proposition 5 to have

$$T_{ij}(x) = \frac{1}{2m^2} \int_B \varphi(x, v) v_{ij} |g|^{\frac{1}{2}} (-\hat{g}^{11})^{-\frac{3}{2}} |\tilde{g}|^{-\frac{1}{2}} d^3v, \quad (\text{A.1})$$

where $v_{ij} = \frac{p_i p_j}{(q^1)^2}$. For $i = j = 2$ (A.1) reads

$$T_{22}(x) = \frac{1}{2m^2} \int_B \varphi(x, v) v_{22} |g|^{\frac{1}{2}} (-\hat{g}^{11})^{-\frac{3}{2}} |\tilde{g}|^{-\frac{1}{2}} d^3v. \quad (\text{A.2})$$

From (3.1) we gain

$$p_2 = \frac{1}{2} g_{12} q^1 (1 + w^2) \text{ on } G^1. \quad (\text{A.3})$$

Thus

$$v_{22} = \frac{1}{4} (g_{12})^2 (1 + w^2)^2 \text{ on } G^1. \quad (\text{A.4})$$

w^2 is now expressed on G^1 in terms of v^A via (2.20) to give

$$w^2 = (\lambda_B^2)^{-1} v^B (-\hat{g}^{11})^{-\frac{1}{2}} - \tilde{g}^{12}. \quad (\text{A.5})$$

The exact expression of (λ_B^A) is needed. Splitting the quadratic form (2.17) yields

$$\begin{aligned} & \hat{g}_{AB} X^A X^B \\ &= \left[(\hat{g}_{22})^{\frac{1}{2}} X^2 + (\hat{g}_{22})^{-\frac{1}{2}} \hat{g}_{2\alpha} X^\alpha \right]^2 \\ &+ \left[(\hat{g}_{33} - (\hat{g}_{22})^{-1} (\hat{g}_{23})^2)^{\frac{1}{2}} X^3 + (\hat{g}_{22} \hat{g}_{34} - \hat{g}_{23} \hat{g}_{24}) (\hat{g}_{22} (\hat{g}_{22} \hat{g}_{33} - (\hat{g}_{23})^2))^{-\frac{1}{2}} X^4 \right]^2 \\ &+ \left[((\hat{g}_{22} \hat{g}_{33} - (\hat{g}_{23})^2) (\hat{g}_{22} \hat{g}_{44} - (\hat{g}_{24})^2) - (\hat{g}_{22} \hat{g}_{34} - \hat{g}_{23} \hat{g}_{24})^2)^{\frac{1}{2}} (\hat{g}_{22} (\hat{g}_{22} \hat{g}_{33} - (\hat{g}_{23})^2))^{-\frac{1}{2}} X^4 \right]^2. \end{aligned} \quad (\text{A.6})$$

By computing \hat{g}_{ij} via tensorial transformation formulae we gain

$$(\hat{g}_{ij}) = \begin{pmatrix} \frac{1}{4} g_{11} + \frac{1}{2} g_{12} + \frac{1}{4} g_{22} & \frac{1}{4} g_{11} - \frac{1}{4} g_{22} & \frac{1}{2} (g_{13} + g_{23}) & \frac{1}{2} (g_{14} + g_{24}) \\ \frac{1}{4} g_{11} - \frac{1}{4} g_{22} & \frac{1}{4} g_{11} - \frac{1}{2} g_{12} + \frac{1}{4} g_{22} & \frac{1}{2} (g_{13} - g_{23}) & \frac{1}{2} (g_{14} - g_{24}) \\ \frac{1}{2} (g_{13} + g_{23}) & \frac{1}{2} (g_{13} - g_{23}) & g_{33} & g_{34} \\ \frac{1}{2} (g_{14} + g_{24}) & \frac{1}{2} (g_{14} - g_{24}) & g_{34} & g_{44} \end{pmatrix}.$$

From (3.1) and (3.2) we get

$$(\hat{g}_{ij}) = \begin{pmatrix} \frac{1}{2} g_{12} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} g_{12} & 0 & 0 \\ 0 & 0 & g_{33} & g_{34} \\ 0 & 0 & g_{34} & g_{44} \end{pmatrix} \text{ on } G^1. \quad (\text{A.7})$$

Hence, in view of (2.17), we gain

$$(\lambda_B^A) = \begin{pmatrix} (-\frac{1}{2} g_{12})^{\frac{1}{2}} & 0 & 0 \\ 0 & (\Omega h_{33})^{\frac{1}{2}} & \Omega h_{34} \\ 0 & 0 & \Omega^{\frac{1}{2}} h_{33}^{-\frac{1}{2}} \end{pmatrix} \text{ on } G^1. \quad (\text{A.8})$$

Thus, the inverse matrix of (λ_B^A) reads

$$(\lambda_B^A)^{-1} = \begin{pmatrix} (-\frac{1}{2} g_{12})^{-\frac{1}{2}} & 0 & 0 \\ 0 & (\Omega h_{33})^{-\frac{1}{2}} & -h_{34} \\ 0 & 0 & \Omega^{-\frac{1}{2}} h_{33}^{\frac{1}{2}} \end{pmatrix} \text{ on } G^1. \quad (\text{A.9})$$

Simple calculation gives

$$\widehat{g}^{11} = 2 (g_{12})^{-1} \text{ on } G^1. \quad (\text{A.10})$$

It is worth noting from (A.10) that the condition $\widehat{g}^{11} < 0$ is equivalent to $g_{12} < 0$ on G^1 . We also have

$$\widetilde{g}^{12} = 0 \text{ on } G^1. \quad (\text{A.11})$$

(A.5), (A.9), (A.10) and (A.11) imply

$$w^2 = v^2 \text{ on } G^1. \quad (\text{A.12})$$

It follows from (A.4) and (A.12) that

$$v_{22} = \frac{1}{4} (g_{12})^2 (1 + v^2)^2 \text{ on } G^1. \quad (\text{A.13})$$

We now handle the term $|g|^{\frac{1}{2}} (-\widehat{g}^{11})^{-\frac{3}{2}} |\widetilde{g}|^{-\frac{1}{2}}$ on G^1 . From (3.1) and (3.2) we gain

$$\det(g_{ij}) = -(\Omega g_{12})^2 \text{ on } G^1. \quad (\text{A.14})$$

Thus

$$|g|^{\frac{1}{2}} = \Omega |g_{12}| = -\Omega g_{12} \text{ on } G^1. \quad (\text{A.15})$$

Using (3.1), (3.2) and tensorial transformation formulae we gain

$$\widehat{g}_{22} = -\frac{1}{2} g_{12}, \quad \widehat{g}_{2\alpha} = 0, \quad \widehat{g}_{\alpha\beta} = \Omega h_{\alpha\beta} \text{ on } G^1.$$

Hence

$$\widetilde{g} = \det(\widehat{g}_{AB}) = -\frac{1}{2} g_{12} \Omega^2 \text{ on } G^1. \quad (\text{A.16})$$

From (A.10), (A.15) and (A.16) we gain

$$|g|^{\frac{1}{2}} (-\widehat{g}^{11})^{-\frac{3}{2}} |\widetilde{g}|^{-\frac{1}{2}} = \frac{1}{2} (g_{12})^2 \text{ on } G^1. \quad (\text{A.17})$$

The insertion of (A.13) and (A.17) into (A.2) gives the desired expression of $T_{22}(x)$.

APPENDIX B: PROOF OF PROPOSITION 7

(3.11) follows from relation (7.50) of [10] by using $g_{1\lambda} = 0$ on G^1 . $T_{2\alpha}(x)$ is computed in the same manner as $T_{22}(x)$. Actually, for $i = 2$ and $j = \alpha$, (A.1) reads

$$T_{2\alpha}(x) = \frac{1}{2m^2} \int_B \varphi(x, v) v_{2\alpha} |g|^{\frac{1}{2}} (-\widehat{g}^{11})^{-\frac{3}{2}} |\widetilde{g}|^{-\frac{1}{2}} d^3 v. \quad (\text{B.1})$$

From (3.1) and (3.2) we gain

$$p_\alpha = q^1 g_{\alpha\beta} w^\beta \text{ on } G^1. \quad (\text{B.2})$$

(A.3) and (B.2) yields

$$v_{2\alpha} = \frac{1}{2} g_{12} (1 + v^2) (g_{\alpha 3} w^3 + g_{\alpha 4} w^4) \text{ on } G^1. \quad (\text{B.3})$$

From (2.20) and (A.9) we gain

$$w^3 = \frac{1}{\sqrt{2}} (-g_{12})^{\frac{1}{2}} \left[(\Omega h_{33})^{-\frac{1}{2}} v^3 - h_{34} v^4 \right], \quad w^4 = \frac{1}{\sqrt{2}} (-g_{12})^{\frac{1}{2}} \Omega^{-\frac{1}{2}} (h_{33})^{\frac{1}{2}} v^4. \quad (B.4)$$

(B.3) and (B.4) imply

$$v_{2\alpha} = -\frac{1}{2\sqrt{2}} (1 + v^2) (-g_{12})^{\frac{3}{2}} \Omega \left[h_{\alpha 3} \left((\Omega h_{33})^{-\frac{1}{2}} v^3 - h_{34} v^4 \right) + h_{\alpha 4} \Omega^{-\frac{1}{2}} (h_{33})^{\frac{1}{2}} v^4 \right] \text{ on } G^1. \quad (B.5)$$

Insertion of (A.17) and (B.5) into (B.1) gives the expression (3.12) of $T_{2\alpha}(x)$.

APPENDIX C: PROOF OF PROPOSITION 9

$T_{\alpha\beta}(x)$ is handled in the same way as T_{22} and $T_{2\alpha}$. From (A.1) we have

$$T_{\alpha\beta}(x) = \frac{1}{2m^2} \int_B \varphi(x, v) v_{\alpha\beta} |g|^{\frac{1}{2}} (-\widehat{g}^{11})^{-\frac{3}{2}} |\widetilde{g}|^{-\frac{1}{2}} d^3 v. \quad (C.1)$$

(B.2) implies

$$v_{\alpha\beta} = g_{\alpha\lambda} g_{\beta\mu} w^\lambda w^\mu \text{ on } G^1. \quad (C.2)$$

Insertion of (A.17) and (C.2) into (C.1) gives

$$T_{\alpha\beta}(x) = \frac{(g_{12})^2 g_{\alpha\lambda} g_{\beta\mu}}{4m^2} \int_B \varphi(x, v) w^\lambda w^\mu d^3 v.$$

Thus

$$\begin{aligned} g^{\alpha\beta} T_{\alpha\beta}(x) &= \frac{(g_{12})^2 \Omega h_{33}}{4m^2} \int_B \varphi(x, v) (w^3)^2 d^3 v + \frac{(g_{12})^2 \Omega h_{34}}{2m^2} \int_B \varphi(x, v) w^3 w^4 d^3 v \\ &\quad + \frac{(g_{12})^2 \Omega h_{44}}{4m^2} \int_B \varphi(x, v) (w^4)^2 d^3 v. \end{aligned} \quad (C.3)$$

The insertion of (B.4) into (C.3) gives the expression (3.24) of $g^{\alpha\beta} T_{\alpha\beta}(x)$ on G^1 .

APPENDIX D: DISCUSSION ON THE INTEGRAL CONSTRAINTS EQUATIONS

The integral system (IS). The integral system (IS) is written as follows

$$\langle \varphi_x, f \rangle = h_1(x), \quad \langle \varphi_x, g \rangle = h_2(x). \quad (D.1)$$

where \langle, \rangle denotes the scalar product in $L^2(B)$ and

$$\begin{aligned} f(v) &= (1 + v^2) v^3, \quad g(v) = (1 + v^2) v^4, \\ h_1 &= \frac{128m^4}{(-g_{12})^7 \Omega} (\psi_3 B_4 - \psi_4 B_3), \quad h_2 = \frac{128m^4}{(-g_{12})^7 \Omega} (\psi_4 A_3 - \psi_3 A_4). \end{aligned} \quad (D.2)$$

As the distribution function must be non-negative, let us seek φ of the form

$$\varphi_x(v) \equiv \varphi(x, v) = [a(x) f(v) + b(x) g(v) + c(x)]^2, \quad (D.3)$$

where $a(x)$, $b(x)$ and $c(x)$ are unknown functions defined on G^1 . Expanding (D.3), we have

$$\varphi = a^2 f^2 + b^2 g^2 + 2abfg + 2acf + 2bcg + c^2, \quad (D.4)$$

where variables have been dropped for simplicity. Let φ be like in (D.4). It holds that

$$\begin{aligned}\langle \varphi_x, f \rangle &= a^2 \langle f^2, f \rangle + b^2 \langle g^2, f \rangle + 2ab \langle fg, f \rangle + 2ac \langle f, f \rangle + 2bc \langle g, f \rangle + c^2 \langle 1, f \rangle, \\ \langle \varphi_x, g \rangle &= a^2 \langle f^2, g \rangle + b^2 \langle g^2, g \rangle + 2ab \langle fg, g \rangle + 2ac \langle f, g \rangle + 2bc \langle g, g \rangle + c^2 \langle 1, g \rangle.\end{aligned}\tag{D.5}$$

Spherical coordinates will be used to calculate each of the following quantities that are needed.

$$\begin{aligned}\langle f^2, f \rangle &= \int_B [(1+v^2)v^3]^3 d^3v, & \langle g^2, f \rangle &= \int_B [(1+v^2)v^4]^2 (1+v^2)v^3 d^3v, \\ \langle fg, f \rangle &= \int_B [(1+v^2)v^3]^2 (1+v^2)v^4 d^3v, & \langle f, f \rangle &= \int_B [(1+v^2)v^3]^2 d^3v, \\ \langle g, f \rangle &= \int_B (1+v^2)^2 v^3 v^4 d^3v, & \langle 1, f \rangle &= \int_B (1+v^2)v^3 d^3v, \\ \langle 1, g \rangle &= \int_B (1+v^2)v^4 d^3v, & \langle g^2, g \rangle &= \int_B [(1+v^2)v^4]^3 d^3v, \\ \langle g, g \rangle &= \int_B [(1+v^2)v^4]^2 d^3v.\end{aligned}\tag{D.6}$$

As B is the open unit ball in \mathbb{R}^3 , we set

$$v^2 = r \cos \theta \cos \lambda, \quad v^3 = r \cos \theta \sin \lambda, \quad v^4 = r \sin \theta, \tag{D.7}$$

with

$$0 \leq r < 1, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \lambda \leq 2\pi. \tag{D.8}$$

Define a domain P in \mathbb{R}^3 as follows

$$P = \left\{ (r, \theta, \lambda) \in \mathbb{R}^3 / 0 \leq r < 1, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \lambda \leq 2\pi \right\}. \tag{D.9}$$

Using the change of variables (D.7), we gain

$$\begin{aligned}\langle f^2, f \rangle &= \int_P f^3 r^2 \cos \theta dP, & \langle g^2, f \rangle &= \int_P g^2 f r^2 \cos \theta dP, \\ \langle fg, f \rangle &= \int_P f^2 g r^2 \cos \theta dP, & \langle f, f \rangle &= \int_P f^2 r^2 \cos \theta dP, \\ \langle g, f \rangle &= \int_P g f r^2 \cos \theta dP, & \langle 1, f \rangle &= \int_P f r^2 \cos \theta dP, \\ \langle 1, g \rangle &= \int_P g r^2 \cos \theta dP, & \langle g^2, g \rangle &= \int_P g^3 r^2 \cos \theta dP, \\ \langle g, g \rangle &= \int_P g^2 r^2 \cos \theta dP,\end{aligned}\tag{D.10}$$

where

$$dP = dr d\theta d\lambda.$$

After expansion and reduction we integrate the above quantities over P to obtain

$$\begin{aligned}\langle f^2, f \rangle &= \langle g^2, f \rangle = \langle f^2, g \rangle = \langle g, f \rangle = \langle 1, f \rangle = \langle 1, g \rangle = \langle g^2, g \rangle = 0, \\ \langle f, f \rangle &= \langle g, g \rangle = \frac{32\pi}{105},\end{aligned}\tag{D.11}$$

From (D.5) and (D.11) it holds that

$$\langle \varphi_x, f \rangle = \frac{64\pi a(x) c(x)}{105}, \quad \langle \varphi_x, g \rangle = \frac{64\pi b(x) c(x)}{105}. \tag{D.12}$$

Thus, φ solves (D.1) if and only if

$$\frac{64\pi a(x) c(x)}{105} = h_1(x), \quad \frac{64\pi b(x) c(x)}{105} = h_2(x). \tag{D.13}$$

In sum φ is given by

$$\begin{aligned} \varphi_x(v) &\equiv \varphi(x, v) = [a(x)(1+v^2)v^3 + b(x)(1+v^2)v^4 + c(x)]^2, \\ x &\in G^1, \quad v = (v^2, v^3, v^4) \in B, \end{aligned} \quad (\text{D.14})$$

where

$$a(x)c(x) = \frac{105h_1(x)}{64\pi}, \quad b(x)c(x) = \frac{105h_2(x)}{64\pi}. \quad (\text{D.15})$$

The integral equation (IE). The integral equation (IE) is written as follows

$$E\langle\varphi_x, e\rangle + I\langle\varphi_x, i\rangle + S\langle\varphi_x, s\rangle = C, \quad (\text{D.16})$$

where

$$\begin{aligned} E(x) &= \frac{(-g_{12})^3}{8m^2}, \quad e(v) = (v^3)^2, \\ I(x) &= \frac{(-g_{12})^3 h_{34}}{4m^2} \left(1 - (\Omega h_{33})^{\frac{1}{2}}\right), \quad i(v) = v^3 v^4, \\ S(x) &= \frac{(-g_{12})^3}{8m^2} \left[(h_{34})^2 (\Omega h_{33})^{\frac{1}{2}} \left((\Omega h_{33})^{\frac{1}{2}} - 2\right) + h_{44} h_{33}\right], \\ s(v) &= (v^4)^2, \quad C(x) = \frac{1}{4} g^{\alpha\beta} (N_{\alpha\beta} + M_{\alpha\beta}). \end{aligned} \quad (\text{D.17})$$

Using the expression of φ given in (D.4) we gain

$$\langle\varphi_x, e\rangle = a^2\langle f^2, e\rangle + b^2\langle g^2, e\rangle + 2ab\langle fg, e\rangle + 2ac\langle f, e\rangle + 2bc\langle g, e\rangle + c^2\langle 1, e\rangle. \quad (\text{D.18})$$

As in the preceding paragraph, the above quantities are found to be

$$\langle f^2, e\rangle = \frac{8\pi}{63}, \quad \langle g^2, e\rangle = \frac{8\pi}{189}, \quad \langle 1, e\rangle = \frac{4\pi}{15}, \quad \langle fg, e\rangle = \langle f, e\rangle = \langle g, e\rangle = 0. \quad (\text{D.19})$$

(D.18) and (D.19) imply

$$\langle\varphi_x, e\rangle = \frac{8\pi}{63}a^2 + \frac{8\pi}{189}b^2 + \frac{4\pi}{15}c^2. \quad (\text{D.20})$$

Similarly it holds that

$$\langle\varphi_x, i\rangle = a^2\langle f^2, i\rangle + b^2\langle g^2, i\rangle + 2ab\langle fg, i\rangle + 2ac\langle f, i\rangle + 2bc\langle g, i\rangle + c^2\langle 1, i\rangle. \quad (\text{D.21})$$

Straightforward calculations as above give

$$\langle f^2, i\rangle = \langle g^2, i\rangle = \langle f, i\rangle = \langle g, i\rangle = \langle 1, i\rangle = 0, \quad \langle fg, i\rangle = \frac{8\pi}{189} \quad (\text{D.22})$$

(D.21) and (D.22) give

$$\langle\varphi_x, i\rangle = \frac{16\pi ab}{189}. \quad (\text{D.23})$$

We are then left with calculating

$$\langle\varphi_x, s\rangle = a^2\langle f^2, s\rangle + b^2\langle g^2, s\rangle + 2ab\langle fg, s\rangle + 2ac\langle f, s\rangle + 2bc\langle g, s\rangle + c^2\langle 1, s\rangle. \quad (\text{D.24})$$

By proceeding as above we get

$$\langle fg, s\rangle = \langle f, s\rangle = \langle g, s\rangle = 0, \quad \langle f^2, s\rangle = \frac{8\pi}{189}, \quad \langle g^2, s\rangle = \frac{8\pi}{63}, \quad \langle 1, s\rangle = \frac{4\pi}{15}. \quad (\text{D.25})$$

(D.24) and (D.25) yield

$$\langle \varphi_x, s \rangle = \frac{8\pi}{189}a^2 + \frac{8\pi}{63}b^2 + \frac{4\pi}{15}c^2. \quad (D.26)$$

From (D.16), (D.20), (D.23) and (D.26), we see that the integral equation (IE) is equivalent to

$$10(3E + S)a^2 + 10(3S + E)b^2 + 20Iab + 63(E + S)c^2 = \frac{945}{4\pi}C. \quad (D.27)$$

In view of (D.15), multiplying (D.27) by c^2 and rearranging, we gain

$$\begin{aligned} & 110\,250(3E(x) + S(x))(h_1(x))^2 + 110\,250(3S(x) + E(x))(h_2(x))^2 \\ & + 220\,500h_1(x)h_2(x)I(x) + 258\,048\pi^2(E(x) + S(x))[c(x)]^4 \\ & = 967\,680\pi C(x)[c(x)]^2. \end{aligned} \quad (D.28)$$

(D.28) is an algebraic equation that can be solved under suitable assumptions to find $c(x)$. Doing so we deduce $a(x)$ and $b(x)$ thanks to (D.15). Finally the distribution function φ is obtained on $\widehat{G^1}$ and has the form (D.14).

Acknowledgement. I am thankful to Professor Marcel Dossa who suggested this topic and guided me throughout the work.

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